Developing Topsy Turvy and Number Planet

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Summary

This article is about a mathematics collaboration in the spirit of Martin Gardner. Igor Kriz illustrated group theory and the “sporadic simple” Mathieu 12 group in Scientific American by turning them into permutation puzzles. He challenged readers to find a mechanical implementation of his M12 puzzle. Oskar van Deventer took up the challenge resulting in two completely different implementations, “Topsy Turvy” and “Number Planet”. This article describes the development of those two implementations.

Challenge in Scientific American

The July 2009 issue of Scientific American featured an article by Igor Kriz on group theory [1]. The purpose of the article was to educate people on group theory. It explained about “simple groups”, which are the group-theory equivalent of prime numbers. It highlighted the multi-decade mathematician’s quest to identify and classify all simple groups. And it illustrated the concept of a “simple sporadic group” with a set of electronic puzzles, programmed by Igor’s graduate student Paul Siegel. One of the puzzles was the “M12” puzzle, based on the simple sporadic Mathieu 12 group.

The M12 puzzle is played as follows. Take twelve tokens, numbered “1” to “12”. There are two permutations: “Invert” and “Merge”, see Figure 1. The object of the M12 puzzle is similar to the object of the Rubik’s Cube: scramble and solve by using only the two permutations.

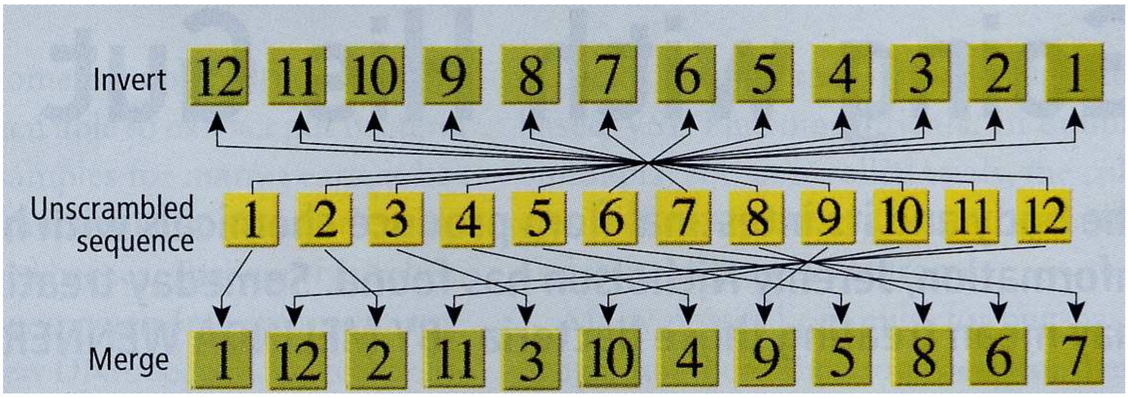

Figure 1: M12 puzzle using two permutations, “Invert” and “Merge”.

Igor suggested in the article that the three puzzles might be mechanically implemented, e.g. using a rotating device and a system of gears, which was left as a challenge to the reader.

Implementation no.1: Topsy Turvy

Oskar van Deventer took up the challenge. The “Invert” operation is easily implemented, as one can just turn around the set of tokens. The “Merge” operation is more challenging. Oskar’s first idea was to implement the “Merge” using some slapstick construction, but he could not find a good mechanism. However the reverse operation, “Split”, might be easier to implement. Oskar contacted Igor, who confirmed that “Split” could be used.

In the recent past, Oskar had developed several puzzle mechanism to manipulate dropping token, “Jukebox” and “Pachinko”, see Figure 2. The former uses physical switches to alternating move tokens left and right. The latter uses a pattern of grooves that holds one or two tokens until one more token is inserted that pushes the other tokens down. Oskar found

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[1] Oskar van Deventer is one of the world’s most prolific designers of mechanical puzzles. Several of his innovative designs are commercially available.

[2] Igor Kriz is Professor of Mathematics at the University of Michigan. His main mathematical interest is algebraic topology.

that the latter mechanism could be used to build the 12-splitters needed for the “Split” operation.

Figure 2: “Jukebox” and “Pachinko” puzzle mechanisms by Oskar

Figure 3 shows a working prototype of “Topsy Turvy”, which implements the 12-splitter. A big crank is used to move the 12 tokens into the 12-splitter. The “Invert” operation is implicitly implemented, as the crank can either turn left or right. When turned, the 12 tokens are dropped in one by one. The first 11 of the token will land stable on top of one another. However, when the 12^{th} token drops, it rolls down over the 11^{th} and while dropping it pushes the 11^{th} out of position. Then the 11^{th} pushes the 10^{th} out of position, which pushes the 9^{th}, which pushes the 8^{th}, etcetera. In a cascade, the whole stack of tokens falls apart, with the even tokens moving to the right and the odd ones to the left.

Figure 3: “Topsy Turvy” featuring a 12-splitter.

Oskar had to build several prototypes to get the mechanism right. The prototypes were built on the laser cutter of Peter Knoppers, using MDF and acrylic. The tokens are made of cast tin, using laser-cut MDF moulds. In order to have the mechanism turn smoothly, four huge gears act as ball bearing, carrying the weight of the crank mechanism. A rattle mechanism is used to force a user to finish a move once started, preventing illegal moves.

The first prototype had the major flaw that a user can continue turning while the tokens are still dropping. If that happens, then tokens will collide when they are caught by the crank mechanism at the bottom, enabling illegal moves and blockage. George Miller found an elegant solution, a toggle switch which limits the rotation of the crank between -240 and +240 degrees. With the switch in place, a user has to turn the crank all the way back which takes sufficient time for the tokens to settle at the bottom. Magnets were used to make the switch bi-stable. Another problem was that tokens could skip the entry if the crank is turned too fast. Peter Knopper’s solution was to place a pin at the top entry of the grooves, which forces the tokens down. With all major problems solved, the third prototype was found to work satisfactory.

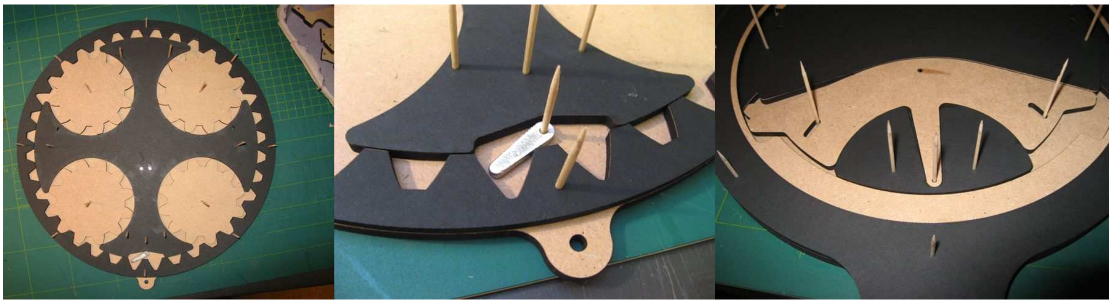

Figure 3: Gears, rattle and toggle switch to prevent illegal moves.

Implementation no.2: Number Planet
While Oskar was working on Topsy Turvy, Igor suggested a completely different implementation. Igor had found a special “planar permutation” that also implements the M12 group. The two permutations are “rotate” and “swap”, see Figure 4. Igor’s idea was to have a rotating mechanism that performs these two permutations.

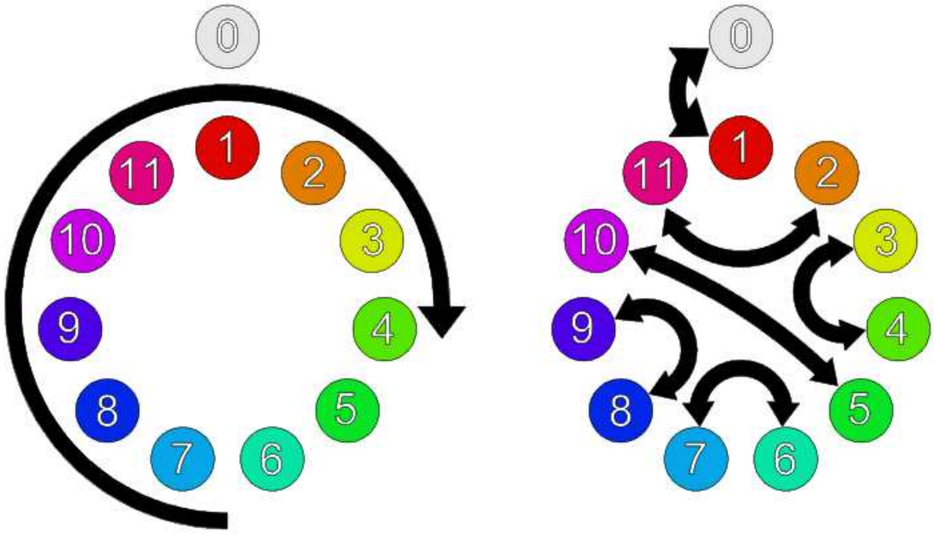

Figure 4: “Rotate” and “Swap” permutations.

Oskar started 3D sketching and after a lot of communication with Igor, they found a mechanism that could do the trick, see Figure 5. Eleven tokens are placed in a circle, enabling the “Rotate” operation. Five disks and rings of varying size are used to perform the pair-wise swaps needed for the “Swap” operation. Notice that the “0” and “1” are not explicitly swapped. The tokens can be in two rows. After making the five turns for a swap, the tokens are in the other row, implicitly swapping “0” and “1”.

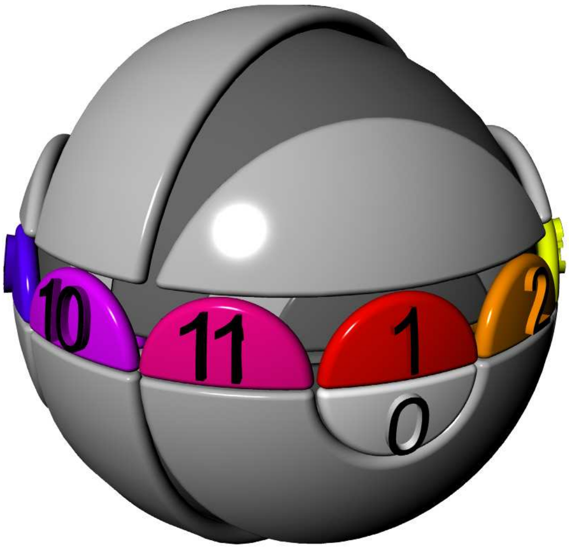

Figure 5: Mechanism implementing “Rotate” and “Swap”.

However, Oskar was not satisfied with the mechanism with the very round tokens. A much better mechanism would be possible, if the “0” and “1” were not surrounded by the “11-2” swap. Oskar asked Igor whether there exist a better planar M12 permutation. Igor started looking using his Maple program, while Oskar used a Python program written by George Miller. With crossing emails, Igor beat Oskar by only five minutes, both discovering that the requested planar permutation does indeed exist. Using this permutation, Oskar made a 3D design using trapezoid token, which push each other better when performing the “Rotate” operation, see Figure 6.

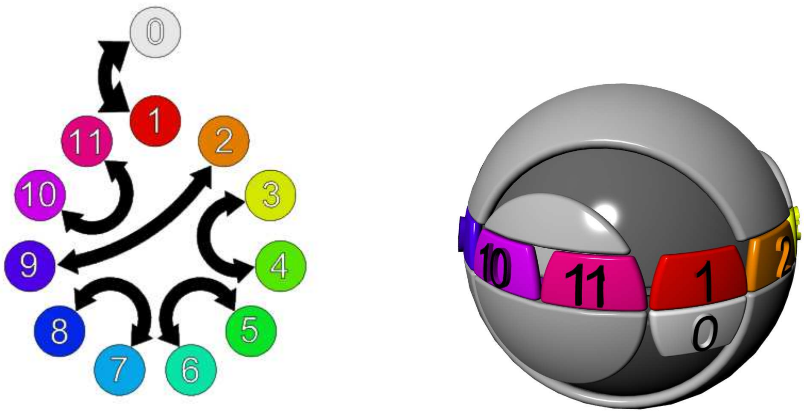

Figure 6: Better planar permutation, enabling trapezoid-shaped tokens.

The puzzle, called “Number Planet” by Igor, was prototyped using 3D printing technologies. The first prototype, made by UM3D in ABS-based Fusion Deposition Modeling, was completely stuck as Oskar had made a modeling error. The second prototype worked reasonably well, but it was still cumbersome, as it’s many subassemblies made it hard to put together. The third prototype, made by TNO Netherlands using nylon-based Selective Laser Sintering and painted with textile acid dye, was finally a success.

Figure 7: “Number Planet” design, and FDM and SLS prototypes.

Solving the M12 puzzles
With working prototypes available, the reader may wonder how these puzzles should be solved. First of all, it should be noted that both implementations feature two permutations, albeit different ones.

Topsy Turvy:
- “Left”: 1-2-3-4-5-6-7-8-9-10-11-12 → 11-9-7-5-3-1-2-4-6-8-10-12.
- “Right”: 1-2-3-4-5-6-7-8-9-10-11-12 → 2-4-6-8-10-12-11-9-7-5-3-1.

Number Planet:
- “Rotate”: 0-1-2-3-4-5-6-7-8-9-10-11 → 0-2-3-4-5-6-7-8-9-10-11-1.
- “Swap”: 0-1-2-3-4-5-6-7-8-9-10-11 → 0-1-9-4-3-6-5-8-7-2-11-10.

Secondly, both mechanisms implement the M12 group, which has 12x11x10x9x8=95040 permutations. The M12 group has the property that if five token are at their correct place, then the other seven tokens are correct too [1]. Using this information one could envision the following solution approaches.

1. God’s Table by computer
   As the solution space is very small, it is quite feasible to build a “God’s Table” enumerating all possible states and the solution sequence for each state. This is exactly what George Miller did. The file that his Python program produced is small enough to be used on a smart phone, so you can always have a solution at hand. Although the God’s Table provides the fastest solution, it is impossible for a human being to memorize.
2. Recursive solution by hand
   A solution worked out by Igor uses a small set of recursive operations. While much easier to memorize, the recursiveness of the solution requires many (thousands?) of moves. This makes the solution rather impractical for the mechanical versions.
3. Computer-aided optimization
   A computer could be used to find the shortest five sets of operations, that bring 5 tokens in place one by one, using the property of M12 mentioned above.

Mission accomplished, now what?
This article presented two completely different implementations for Igor Kriz’ M12 puzzle challenge. Topsy Turvy uses gravity and its operations are non-reversible. Number Planet is a twisty puzzle and every operation can be undone. Topsy Turvy occasionally cascades too early if it is handled too wildly. The prototypes are not perfect. The rough surfaces of the 3D-printed Number Planet prototypes are sometimes a bit sticky.

Both puzzles are excellent illustrations of M12 “simple sporadic group”.They would make interesting collector’s items for connoisseurs. At the moment of writing this article, it is unclear whether the puzzles have any further commercial potential.

References

[1] Igor Kriz, Paul Siegel, ”Simple Groups at Play”, Scientific American, Vol.299, No.1, pp.64-89, July 2008.